Choosing a charge law for four dimensions: what 1/r does to atoms and the periodic table

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Abstract

Gauss's law gives charges in four spatial dimensions a $1/r^2$ potential, under which no atom has a ground state. We ask what atoms exist if the law between charges is instead pinned at $1/r$, the only power law under which orbitals of different angular momentum share a shell. In four dimensions $1/r$ is not harmonic, and its price is the shell theorem: an inner cloud screens the nucleus only partly, and a neutral atom is felt from outside. We solve every neutral atom through $Z = 140$ self-consistently with local exchange, cross-checked by numerical Hartree–Fock, each method tested first on our own world. The table has rows of 2, 10, 10, 28, 28, 60 elements and noble gases at 2, 12, 22, 50, 78, 138, as the generalized Madelung rule predicts, and its larger subshells make room for 52 elements with no counterpart in ours. Solving it again with the shell theorem forced back on shows what that loss decides: the rows survive it, but what fills inside them from the first d element on does not; solving it under $1/r^{0.5}$ and $1/r^{1.5}$ moves the noble gases, so the rows themselves need the exponent 1/r. Partial screening holds outer electrons a median 2.54 times harder, so the hardest-held electron belongs to a full d shell rather than a noble gas; closed-shell atoms attract with a permanent $-\frac{3Q_AQ_B}{64R^5}$ that outranges dispersion; and four-dimensional space breaks Hund's first rule for d^2, d^{16}, $4f^3$ and $4f^{29}$.

1 · Introduction

Why does space have three dimensions? Ehrenfest (1918, 1920) turned the question into physics. He asked what in the laws of physics would change with the number of dimensions, and found, among other things, that under the inverse-power force Gauss's law gives in D dimensions, no planetary orbit is stable for $D > 3$. Tangherlini (1963) carried the argument into general relativity and into quantum mechanics: the hydrogen atom with Gauss's potential has no stable states above three dimensions. The usual conclusion is that a world with more than three large dimensions has no atoms. Callender (2005) points out what that conclusion leans on, that the laws carry over to other dimensions unchanged.

They needn't. The literature on the hydrogen atom in D dimensions (Nieto 1979; Al-Jaber 1998) and on dimensional scaling (Herschbach, Avery, and Goscinski 1993; Loeser and Herschbach 1986a, 1986b; Ghosh, Kais, and Herschbach 2020) keeps the potential at $1/r$ in every D, mostly as a convention that keeps the problem solvable. Négadi and Kibler (1996) used the D-dimensional hydrogen atom to extend Madelung's filling rule (Madelung 1936) to any dimension; applied to four, it gives rows of 2, 10, 10, 28, 28, 60. Pyykkö and Zhao (1991) solved the atoms of a two-dimensional world by Hartree–Fock, after a table from hydrogen's filling (Asturias and Aragón 1985). Above three dimensions there are self-consistent atoms, but not a table: Hartree–Fock for two-electron atoms in any D (Loeser and Herschbach 1986a), the Thomas–Fermi atom in n dimensions (Kventsel and Katriel 1981), and many-electron models from the limit of large D (Loeser 1987; Loeser et al. 1994). To our knowledge this paper is the first shell-resolved self-consistent survey of the neutral atoms through $Z = 140$ in four Euclidean dimensions, for a pairwise $1/r$ interaction and electrons with two spin states, with local exchange, no correla-

tion and a first-order correction for open subshells' terms, cross-checked by Hartree–Fock. For hydrogen, Burgbacher, Lämmerzahl, and Macias (1999) set Gauss's law against $1/r$ in D dimensions; to our knowledge nobody has asked what the choice of $1/r$ does to atoms with many electrons.

This paper does both. Part I treats the law as a choice: why one has to be made, why $1/r$ exactly, and what it costs in four dimensions (section 2). Part II solves every neutral atom to $Z = 140$ under it (sections 3 and 4) and finds that four-dimensional space breaks Hund's first rule (section 5). Part III solves the table again with the shell theorem forced back on, to find which of its features the law decides (section 6), and gives the pull between neutral atoms that later work on molecules starts from (section 7). The appendices, in the supplement, hold the arguments and methods in full, every check of every method against our own world, and the starting data for chemistry: valence states, the atoms' response, and a test of the valence rule. Every number in the text is recomputed from the code and its data whenever anything changes, and checked against a ledger of 42 claims. The solvers, with their known-answer tests and a script that solves the paper's atoms again, are released under the MIT licence at tangled.org/morphyxmino.bsky.social/four-atom; the data and the paper's making are public at four.mino.mobi/papers.

Part I · The law

2 · The law between charges

Ask what the atoms are in a world with four dimensions of space and there is no answer yet, because the question leaves out the law between charges. In ours that law isn't a free choice. Gauss's law, which a local, linear field theory of charge gives, spreads a point charge's field over the sphere around it, whose area grows as r^{D-1}, so the potential falls as $r^{-(D-2)}$. In three dimensions that is $1/r$. In four it is $1/r^2$, and with $1/r^2$ there are no atoms.

2.1 Gauss's law leaves no atom. One electron in the pull $V = -\alpha/r^2$ has the Hamiltonian $H = -\frac{1}{2}\nabla^2 - \alpha/r^2$, and nothing in it is a length. Stretch a wavefunction, $\psi(x) \to \psi(\lambda x)$, and both terms scale by λ^2, so a bound level at $E < 0$ would bring one at $\lambda^2 E$ for every λ. Either nothing binds or the levels run down without end. Hardy's inequality (Barbatis 2024) decides which: for $\alpha \le \frac{1}{2}$ the kinetic energy always outweighs the pull and nothing binds, and for $\alpha > \frac{1}{2}$ the wavefunction changes sign infinitely often on the way in and the electron falls to the centre (Case 1950; Landau and Lifshitz 1977, sec. 35) (appendix F.1). Our one-electron solver shows both (figure 1a). For $\alpha = 0.45$ it finds no level on any grid. For $\alpha = 0.7$ the lowest level is -5.87×10^{3}, -5.87×10^{5} and -5.87×10^{7} hartree as the grid's inner end moves from 10^{-3} to 10^{-5} bohr: set by where the calculation stops, not by the electron. There is no Bohr radius, and classically no stable orbit either (Ehrenfest 1918).

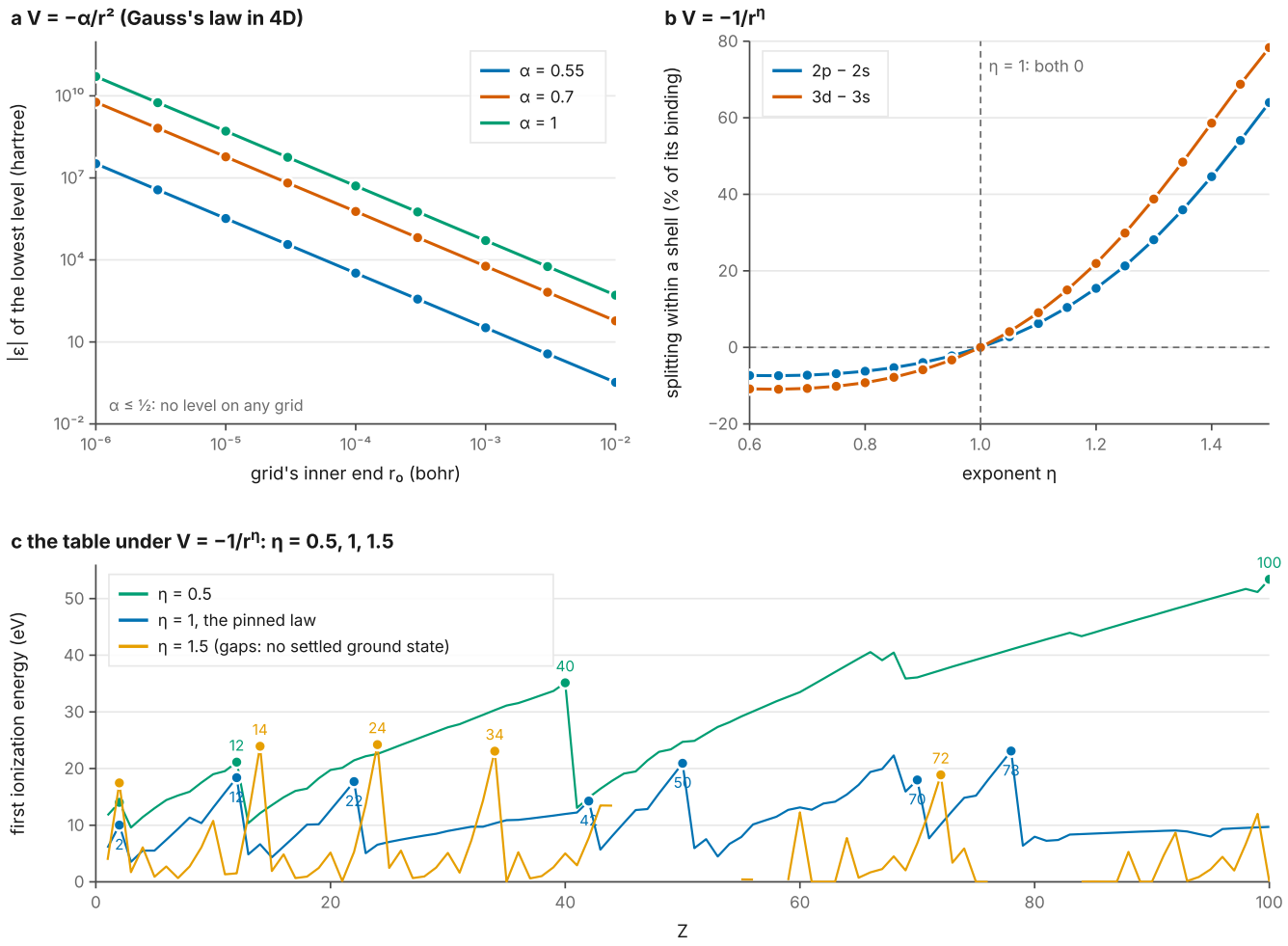


Figure 1. The choice of law. (a) Gauss's law in four dimensions gives a potential $-\alpha/r^2$. One electron in it, solved on grids whose inner end r_0 is moved in: for every α above ½ the lowest level scales as $1/r_0^2$, set by where the calculation stops rather than by the electron (the fall to the centre); at or below ½ there is no level on any grid (Hardy's inequality). Either way there is no Bohr radius and no atom. (b) With a pull $-1/r^\eta$ instead, the splitting between 2p and 2s (and 3d and 3s) as a fraction of the shell's binding vanishes only at $\eta = 1$, to 10^{-11}: the hidden SO(5) symmetry of 1/r, which no other exponent has (section 2.3). (c) The whole table, elements 1–100, solved under each exponent (every pair of charges alike; appendix F.8): first ionization against Z, each law's noble gases marked by one rule (a closed configuration, a local maximum, the next atom at least 30 % lower and at least half as hard as any before). Under $\eta = 0.5$ they fall at hydrogen's shells, 2, 12, 40, 100; under $\eta = 1.5$ at 2, 14, 24, 34, 72, with 18 atoms unsettled; only the pinned law gives the generalized Madelung rule's rows (2, 12, 22, 50, 78, with the closed d-and-s atoms 42 and 70, zinc's kind).

2.2 The way out taken. Something has to give: the squeeze $\frac{\hbar^2}{2m_e r^2}$ that resists the pull, or the pull. An electron whose energy grows faster than p^2 binds under Gauss's law, at the cost of Galilean relativity for all matter. A small fourth dimension gives back three-dimensional atoms. This universe keeps the electron and four large dimensions and changes the law: the potential between charges falls as $1/r$. The choice between the first route and this one is a judgement, stated as one (appendix F.2): the electron route keeps our Faraday cages, the law route keeps our mechanics.

2.3 Only 1/r keeps a shell together. Any pull $-1/r^\eta$ with $0 < \eta < 2$ binds and has a length of its own. What singles out $\eta = 1$ is a symmetry. Under $1/r$ every bound orbit is a closed ellipse (Santos, Soares, and Tort 2007), so the Runge–Lenz vector $A_i = \sum_j L_{ij}\, p_j - Z x_i / r$ is conserved, and in quantum mechanics the rotations SO(D) together with **A** generate SO($D + 1$) (Fock 1935; Alliluev 1958; Bander and Itzykson 1966). A level of hydrogen is then the set of degree-($n - 1$) harmon-

ics on a sphere one dimension up, so every l below n shares the energy, in four dimensions $-\frac{Z^2}{2(n+\frac{1}{2})^2}$. Three things should be kept apart here. That each l has $(l+1)^2$ orbitals is the four-dimensional rotations' doing, in any central potential. That every l below n shares one energy is $1/r$'s, through SO(5); together they make hydrogen's shells of 2, 10, 28 and 60 electrons, each a group of levels with a gap above it. The table's rows are a third thing, many electrons filling subshells in an order only a calculation finds (section 4.1). Without the symmetry nothing holds the l together. On the same solver 2p sits −6.2 % from 2s at $\eta = 0.8$, 5×10^{-10} % at $\eta = 1$, +15.4 % at 1.2 and +64.0 % at 1.5, as a fraction of the shell's binding (figure 1b). Rows would still form under another exponent, but nothing guarantees they would be the same, and the generalized Madelung rule, which starts from hydrogen's degeneracy, wouldn't apply (appendix F.3).

2.4 What 1/r costs: the shell theorem. In four dimensions $\nabla^2(1/r) = -1/r^3$ away from the charge, so there is no Gauss's law for it: the flux through a sphere round a charge grows with the sphere (appendix F.4). A thin spherical shell of charge with radius s, felt at radius r, gives $\kappa(t)/\max(r,s)$, $t = \min(r,s)/\max(r,s)$, with κ the law's $1/|x-y|$ averaged over the shell. In ours $\kappa = 1$, the shell theorem, because $1/r$ is harmonic and a harmonic function averages over a sphere to its value at the centre; in flat space of any dimension the only potentials with one are Gauss's, plus at most r^2 and a constant (Tse, Markowich, and Phan 2026; Landkof 1972). Here the Laplacian is negative, so $\kappa < 1$: it falls from 1 to $\kappa(1) = \frac{8}{3\pi} = 0.848826$ at the shell (appendix A, figure A1). Three things follow. An inner cloud screens the nucleus from an outer electron only partly. A hollow conductor's centre sits at $3\pi/8 = 1.178097$ of its wall's potential, so there is no Faraday cage. And a neutral atom is felt from outside, as $-\frac{Q}{8R^3}$ at a distance R, with $Q = \sum \langle r^2 \rangle$ over its electrons (section 7). Everything distinctive in Part III comes from here.

2.5 Where 1/r comes from, and what is declared. A potential $1/r^\eta$ in D dimensions is the Fourier transform of $1/k^{D-\eta}$, so the pinned law needs $1/k^3$ where Maxwell's statics give $1/k^2$. No local field equation gives it. Of the five routes we examined (appendix F.5), the one that survives is a vacuum whose permittivity grows with wavenumber, $\varepsilon_0\, \ell\, k$ with ℓ a length (Tarasov and Trujillo 2013). With the permeability $\mu_0/(\ell\, k)$, the fields $\mathbf{D}/\varepsilon_0$ and $\mu_0\mathbf{H}$ obey four-dimensional Maxwell's equations exactly, and what pushes a charge is that light seen through a lens, $\mathbf{E} = (\mathbf{D}/\varepsilon_0)/(\ell\, k)$; we call it the **lens vacuum**. Its price is relativity, for the force between charges only: the static $1/r$ acts across space at once. This paper uses only its statics, which are all an atom's ground state needs. The free electron keeps its own relativity, which the lens vacuum gives up only for the force between charges, and that fixes how many electrons an orbital holds. In 4 + 1 dimensions the smallest Dirac electron has four components, the product of the four spatial Dirac matrices serves as time's, and a slow electron keeps the two components of one chirality, which turn with one half of a four-dimensional rotation only: two spin states, two to an orbital (appendix F.6). A four-state electron is possible in principle and would double every shell. The same relativity of the free electron gives Pauli's exclusion. The half of the spin-statistics theorem that makes half-integer spin a fermion needs only a Dirac equation and an energy bounded below, not the locality the force here lacks: electrons that could share a state would have no ground state, the pinned $1/r$ between them notwithstanding (appendix F.6). What stays declared is that the electron is that smallest spinor. Nuclei are point charges stable to $Z = 140$, and the fine-structure constant is small, so atoms are solved without relativity. Single atoms don't collapse (their energies follow the Thomas–Fermi $Z^{5/2}$ (Kventsel and Katriel 1981), fitted exponent 2.527), but the stability of matter in bulk, whose proofs in ours lean on the shell theorem, is open (appendix F.7).

Part II · The table

3 · Methods

Everything here is computed by code written for this paper, JavaScript with no outside libraries; nothing is taken from a quantum-chemistry package, and two independent codes check it, Hartree–Fock in a Gaussian basis for any D (McMurchie

and Davidson 1978) and, in ours, PySCF (Sun et al. 2020) (appendix E.2). An atom is a point nucleus of charge Z and Z electrons, non-relativistic, with every pair of charges interacting through $q_1 q_2 / r$. Each electron moves in the direction-averaged field of the nucleus and the other electrons' clouds, so the problem is radial. With $\psi = r^{-(D-1)/2}\, u(r)\, Y$, Y a harmonic of degree l on the $(D-1)$-sphere,

$$-\tfrac{1}{2}\, u'' + \left[\frac{L(L+1)}{2r^2} + V(r)\right] u = \varepsilon\, u, \qquad L = l + \frac{D-3}{2},$$

so four dimensions differ from ours by the half-integer shift of L and by the degeneracies, $2(l+1)^2$ electrons to a subshell. Four-dimensional hydrogen's ground state is −2/9 hartree; the solver gives −0.22222. Every electron–electron term goes through the kernel κ of section 2.4, which is where the missing shell theorem enters the atom. Exchange is local, from the uniform electron gas of the same dimension and law, $\varepsilon_x = -A_D\, \rho^{1/D}$ with $A_4 = \frac{16}{15\sqrt{\pi}}$ (Loos and Gill 2016) (appendix B), each electron's own share removed (Perdew and Zunger 1981); there is no correlation term.

The radial equation is solved by Numerov's method on a logarithmic grid, each level by Sturm bisection, and the field iterated to self-consistency. The ground configuration is searched, not assumed: from the hydrogenic filling, refilled by the orbital energies, then every move of one or two electrons among the open subshells, a candidate counting only if its field settles. First ionization energies are differences of two self-consistent energies, the ion searched for its own ground state. Hund's first rule, which the search assumes, is checked on every open subshell by diagonalizing its terms, and the table is corrected where it fails (section 5). Two further calculations test how much of the table depends on the method and on the law. The counterfactual solves every element again with the shell theorem forced on, $\kappa \to 1$, everything else unchanged (section 6). Numerical Hartree–Fock, with exact exchange, ranks the same candidates a second way; its agreement with local exchange, the candidates it covers and its margins are reported separately (section 4.3, appendix E.6). The grid, the stop and the quadratures are converged well past what the text quotes (appendix C), every method has a known-answer test run on every change, and appendix G gives the methods in full.

4 · The table

4.1 Rows and noble gases. The table's rows hold 2, 10, 10, 28, 28, 60 elements (and 2 of the next by element 140), and its noble gases are elements 2, 12, 22, 50, 78, 138, each a peak of the first ionization energy (figures 2 and 3). Those are the rows the generalized Madelung rule of Négadi and Kibler (1996) gives when applied to four dimensions (they tabulate two and three); to our knowledge this is the first self-consistent table that confirms them. The rule is a filling order, and like Madelung's in ours it doesn't promise each atom's ground configuration. Read that way, 103 of the 140 are exactly the rule's. Each of the other 37 moves one or two electrons between two subshells, 34 of them from the outer s into the d or f below it, as our chromium, copper and palladium do. The solver finds more of these departures than there are: over our first 36 elements it makes 6 where the measured configurations make 2, from its lean toward d (appendix E.3). So the 4D count is likely too high, and these departures, not the rows, are where the table is least sure.

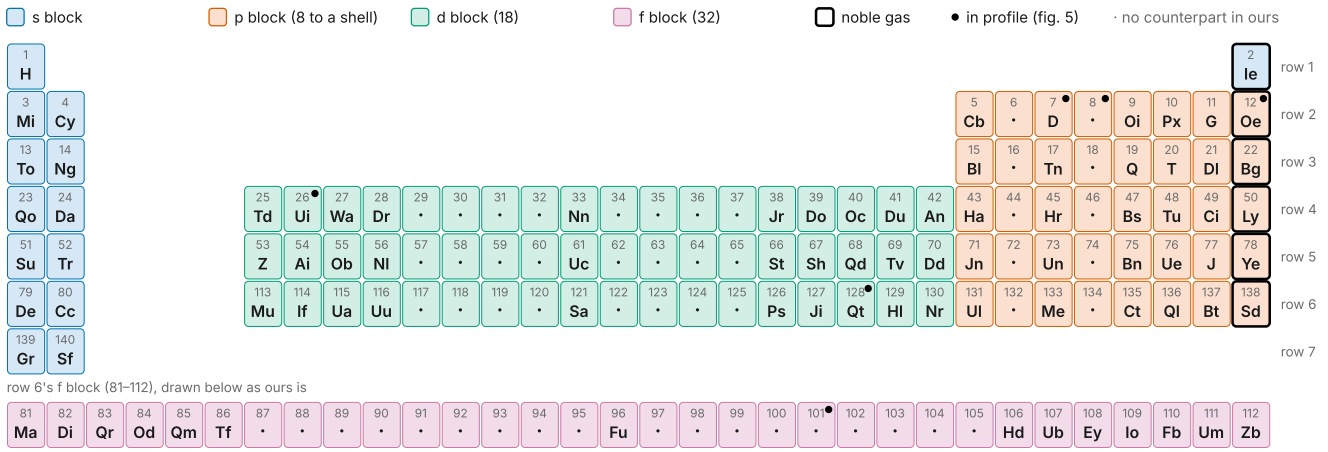


Figure 2. The periodic table of four dimensions, Z = 1 to 140, each atom solved self-consistently under the pinned 1/r. Rows close at the noble gases 2, 12, 22, 50, 78 and 138, and hold 2, 10, 10, 28, 28 and 60 elements, as Négadi and Kibler's generalized Madelung rule gives when applied to four dimensions. A subshell holds $2(l + 1)^2$ electrons, so the p block is 8 wide, the d block 18 and row 6's f block 32 (drawn below, as ours is). An element takes our element's name with its first letter moved on where its place answers to one of ours; the rest (·) have no counterpart. Dots: the elements in figure 4. Hover a cell for its configuration and ionization.

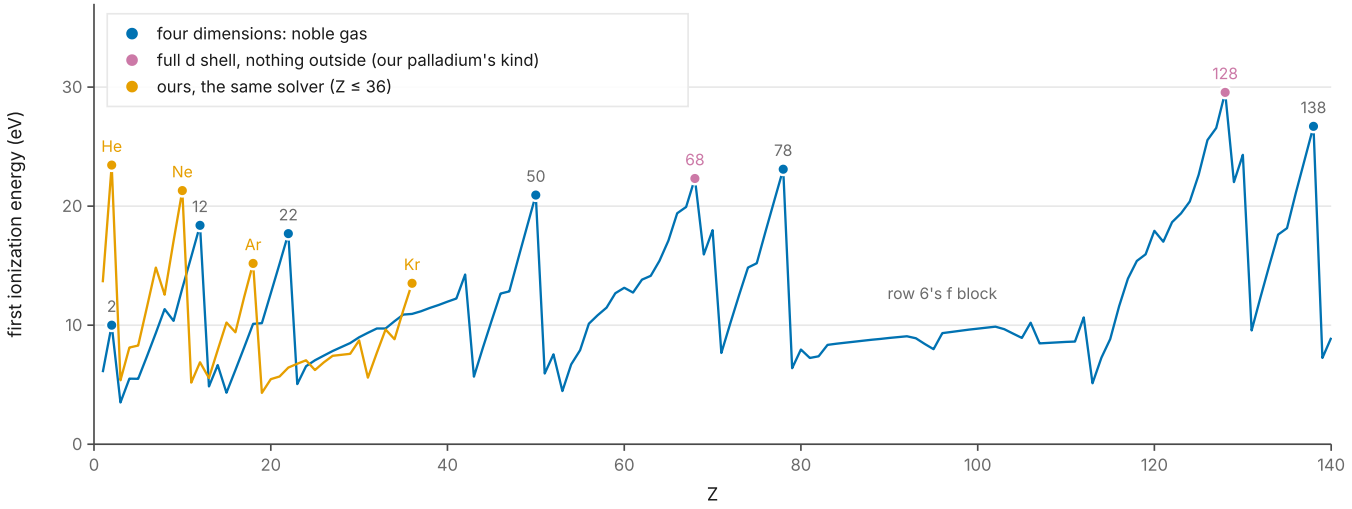


Figure 3. First ionization energy against Z across the four-dimensional table. The noble gases close their rows, but the hardest-held electron isn't a noble gas's: element 128, a full 18-electron d shell with nothing outside it (our palladium's kind), holds its outermost electron at 29.6 eV against its row's noble gas at 26.7; element 68, the same kind a row up, comes within a volt of its row's. In ours palladium holds far less hard than xenon. Across row 6's 32-wide f block (81–112) the binding barely moves. Our own world from the same solver, Z = 1–36, is drawn for scale (orange): its noble gases at 23.4 (He), 21.3 (Ne), 15.2 (Ar) and 13.5 (Kr), against 24.6, 21.6, 15.8 and 14.0 eV measured.

4.2 The hardest-held electron isn't a noble gas's. Element 128 has an 18-electron d shell, full, with nothing outside it. It holds its outermost electron at 29.56 eV, harder than its row's noble gas 138 (26.71 eV) and harder than any other atom in the table. Element 68, the same kind a row up, comes within a volt of its noble gas 78 (22.32 eV against 23.10). Ours goes the other way: palladium's full $4d^{10}$ gives up an electron at 8.3 eV, xenon at 12.1 (Sansonetti and Martin 2005). Searching each ion for its own ground state leaves 128's ionization where it was. Section 6 shows the law does it.

4.3 How firm each configuration is. A second method, numerical Hartree–Fock, ranks the same candidates (appendix E.6). Agreement, the search's coverage and the margins are reported separately. Hartree–Fock and local exchange agree on

126 of the 140 ground configurations and differ on 14: 30, 31, 37, 38, 40, 58, 61 in the d rows, where local exchange puts one electron more in d, as it does in ours; 107, 108, 109, 110, 111 at the end of row 6's f block, where it puts one more in f; and 2 at the start of a block (53, 83). In ours Hartree–Fock gets 36 of 36 configurations right against local exchange's 32, and where the two agree they are right. Agreement is not a guarantee, though: in our lanthanides both overbind f, Hartree–Fock by up to 4.1 eV in the measured states. Taking that as a sensitivity scale (an assumption, not a bound), 17 of the 32 elements of row 6's f block have no competitor with fewer f within it, 5 agree with one inside it, and 4 agree with none solved. The interactive table gives each element's group.

4.4 Every element in profile. Figure 4 draws six atoms' radial densities, and figure 5 what their outermost orbitals look like on the 3-sphere. The same two pictures for every element are on the interactive table, where any subshell can be cut by any of the six planes of four-dimensional space (four.mino.mobi/table; `#Z` opens on element Z). The configurations, energies, ionizations and radii are in `data/table4.csv`. What chemistry starts from, each atom's valence states and a direct test of the valence rule on darbon's hydride DH_5, is in appendix I.

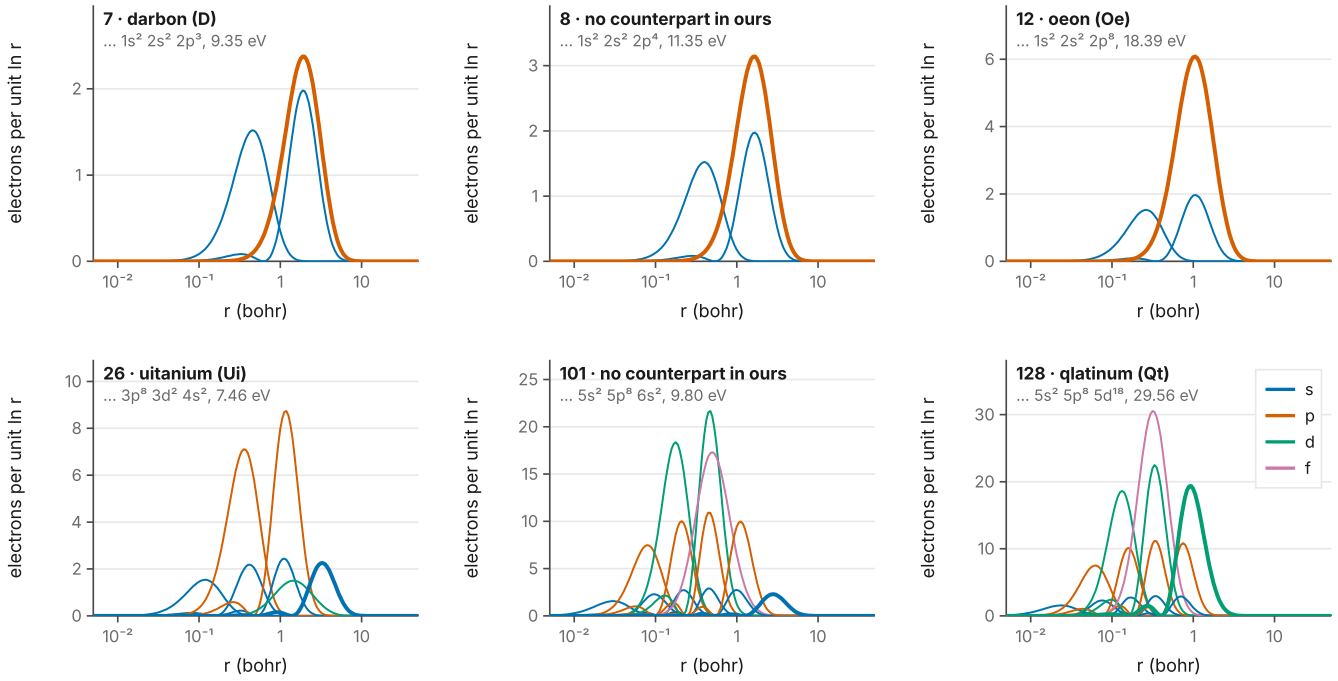


Figure 4. Six four-dimensional atoms in profile: each subshell's electrons per unit ln r against distance, from the solved atoms; the area under a curve is its electron count, and the outermost subshell is drawn thick. Darbon (7), our carbon's place, five electrons of a ten-electron shell; element 8, which answers to nothing of ours; oeon (12), the first noble gas past helium, its p shell holding eight; uitanium (26), where the d block starts; element 101, 21 electrons into an f shell that holds 32; and element 128, the hardest-held electron in the table. Every element's profile: four.mino.mobi/table/#Z.

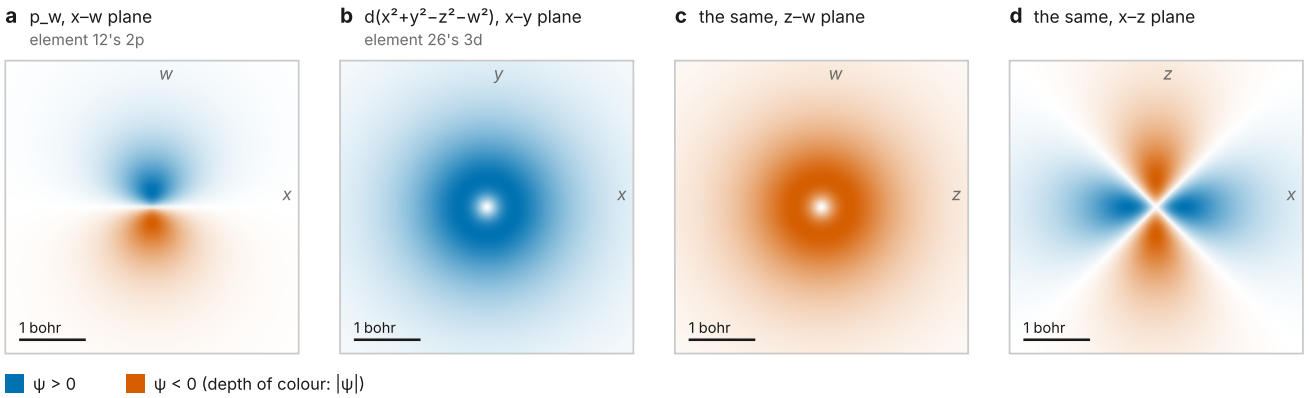


Figure 5. Orbitals four dimensions has and ours doesn't, cut through the nucleus by a plane, on the solved atoms' own radial functions. (a) The fourth p orbital, p_w, in the x–w plane: a dumbbell along w, which vanishes on the slice w = 0, so the three-dimensional slice through the nucleus that is our own space misses it entirely; a slice tilted into w meets it. (b–d) One of the nine 4D d orbitals, $x^2 + y^2 - z^2 - w^2$, in three planes: positive all the way round the x–y plane, negative all the way round the z–w plane, and a four-leaf clover in a plane that mixes them. The x–y and z–w planes are completely perpendicular and meet only at the nucleus; nothing in three dimensions is positive round one plane and negative round another. Any element's orbitals, in any plane: four.mino.mobi/table/#Z; hydrogen's exact ones, turned live: kata.mino.mobi/orbital.

5 · Hund's first rule in four dimensions

The solver gives each open subshell the most parallel spins it can hold. That is Hund's first rule, the known answer for the ground terms of free atoms in ours. In four dimensions it is an assumption, and for some fillings it is wrong. To test it, every arrangement of each open subshell's electrons is diagonalized under the law, on the atom's own orbital (appendix H.1). In ours this reproduces Racah's term separations to the wavenumber, and maximum spin is lowest in all 23 open subshells of our table.

5.1 d^2, d^{16}, $4f^3$ and $4f^{29}$ break the rule. In four dimensions all 36 open p subshells keep the rule, and so do the d fillings d^1, d^3–d^{12}, d^{14}, d^{15}, d^{17}. The exceptions are d^2 and its mirror d^{16}, in all three d rows: element 26 ($3d^2$, 0.40 eV), element 39 ($3d^{16}$, 0.77 eV), element 54 ($4d^2$, 0.30 eV), element 66 ($4d^{16}$, 0.58 eV), element 114 ($5d^2$, 0.36 eV), element 126 ($5d^{16}$, 0.63 eV). In each the ground term is a spin singlet of ten states, and the triplet the solver assumed lies 0.30–0.77 eV above it. In the f block $4f^3$ (element 83) fails too, and so does its mirror $4f^{29}$ (element 108), each with a ground term of spin ½, 0.15 and 0.26 eV below the quartet. No other f filling was found to break the rule, though for the fullest blocks only the highest spins could be run (appendix H.2).

5.2 For d^2, the four-dimensional shell breaks it, not the law. On element 26's own orbital, held fixed, Gauss's $1/r^2$ gives the same order, the singlet 0.82 eV below the triplet against 0.40 eV under $1/r$. The cause is the shell. For two electrons in a d subshell each term's energy is exactly $F^0 + a_2F^2 + a_4F^4$: the F^n are the radial parts of the repulsion and depend on the orbital, and the $a_n = \langle C_n(\cos\gamma)\rangle$ over the term depend only on the term (γ the angle between the two electrons seen from the nucleus). The ten-state singlet, $(2,0)\oplus(0,2)$ of SO(4), holds the electrons nearest right angles, $\langle\cos^2\gamma\rangle = 0.125$ against random's 0.250 and the best triplet's 0.188, and lies below every triplet whenever $F^4/F^2 < 5/3$. Element 26's orbital has 0.39 under the law and 0.65 under Gauss's, so the rule breaks either way. In ours the triplet ^{3}F is the term that keeps the electrons nearest right angles (see Morgan and Kutzelnigg 1993, on Hund's rules and the angles between electrons), and it is the lowest for every F^4/F^2. Pauli keeps same-spin electrons apart, which is the usual reason for Hund's rule; four-dimensional space gives paired spins a better way. Every four-dimensional rotation splits into a left and a right half, a d orbital carries one unit of each, and the singlet adds the two electrons' left halves and cancels their right ones, or the other way round. Three-dimensional rotations don't split, so ours has no such state (appendix H.3; interactive at four.mino.mobi/atom/hund.html).

5.3 The correction moves energies and ionizations, no configuration. Where the rule fails, the solver's energy is the maximum-spin state's, too high by the gap. Every d- and f-block element was searched again with each candidate's gaps added (appendix H.4). **The terms change no ground configuration.** They lower the eight Hund-breaking atoms' energies and move first ionizations in pairs, by 0.30–0.67 eV: up for an atom whose own ground term gains the gap, down for the next, whose ion lands on the broken filling. The wider search found one configuration the bake had missed, element 82's $4f^1\,5d^1\,6s^2$, our cerium's kind. The table of section 4 is the corrected one (`data/table4.csv`); the Hund-spin table is supplement S1. In a ligand field the singlet survives on the 16-cell's eight axes and gives way to a triplet, barely, at a tesseract's sixteen corners (appendix H.5).

Part III · What the law does

6 · What the law decides

To see which of the table's features the law decides, every 4D element was solved a second time with one thing changed: the clouds' potential taken as if the shell theorem held ($\kappa = 1$), and everything else, the four-dimensional kinetic energy, the degeneracies, exchange, the search, left as it was. Each search was seeded with the real table's configuration, and the real law was re-solved in every counterfactual configuration, so a difference is the law's, not a weaker search (appendix G.10). Of the 140 elements, 112 settle, every one up to element 88; the rest, 89–116, have no settled ground state under the shell theorem from any start tried, so what follows about the rows holds through row 5.

6.1 The rows survive the shell theorem. With the shell theorem forced on, the first 24 elements keep every configuration, and the noble gases close the same rows at elements 2, 12, 22, 50, 78, 138 (figure 6). So the rows don't come from the shell theorem's failure. Their lengths, 2, 10, 10, 28, 28, are sums of subshells of $2(l + 1)^2$, filled in the order these calculations find under $1/r$; whether they survive a change of the exponent is a separate question.

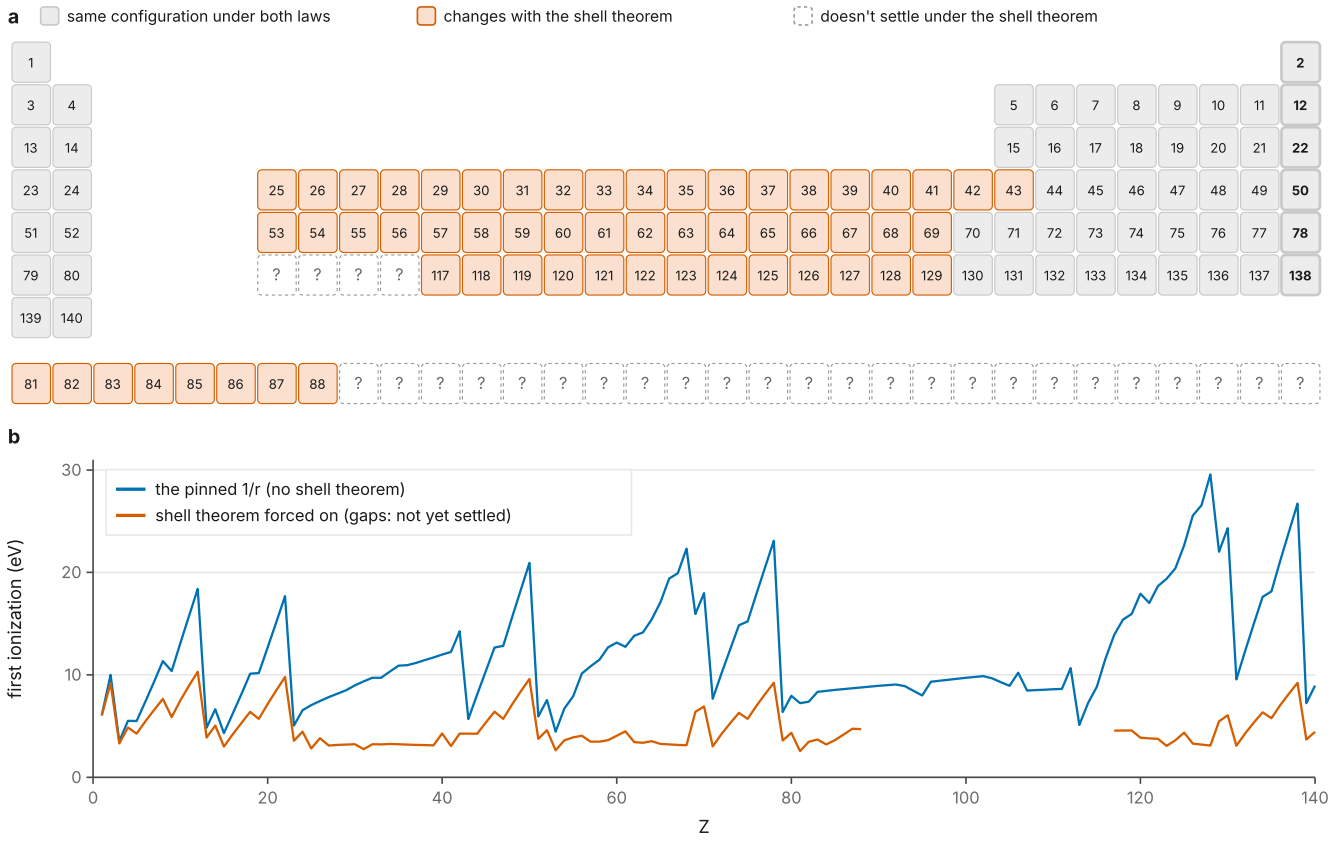


Figure 6. What the law does. Every 4D atom solved a second time with the shell theorem forced on (the clouds' pull as if each sat at the centre), everything else held. (a) The table: grey, the same ground configuration under both laws; orange, changed; a question mark in place of the element's number, no settled ground state under the shell theorem (row 6's f block, 89–116, after a second, wider search). Every atom up to 88 settles. The first 24 elements and all six noble gases are unchanged: the rows survive the shell theorem, at least through row 5. Every change starts at the first d element: without the law, p beats d inside each row. (b) First ionization under both: the law holds outer electrons a median 2.5 times harder, and element 128 nearly ten times (29.6 eV against 3.1); with the shell theorem the noble gases hold hardest, as ours do.

6.2 What fills inside the rows is the law's. Of the settled atoms, 57 change configuration, the first at element 25, the first d element. Under the shell theorem p beats d: element 25 takes $4s^2\ 4p^1$ where the law gives $3d^1\ 4s^2$, and the change runs on through the d and f blocks.

6.3 Partial screening holds outer electrons harder. In ours an inner cloud screens the nucleus from an outer electron whole. Here it screens only partly, by $\kappa(t)$, so outer electrons see more of the nucleus. Across the 110 atoms past helium that settle under both, the law holds the outermost electron a median 2.54 times as hard as the shell theorem would, from 1.06 to 9.55; among the noble gases the ratio grows from 1.09 (helium) to 2.90 (element 138). The largest is element 128's: its full $5d^{18}$ shell holds its electron at 29.56 eV under the law and at 3.1 eV under the shell theorem, where the noble gases are again the hardest-held atoms, as ours are. The table's hardest-held electron is partial screening at its strongest, a full shell whose own cloud hides the nucleus from it least.

6.4 Negative ions. Partial screening holds an extra electron harder too. One electron short of a full p shell (p^7 in four dimensions), atoms bind an extra electron by 3.4, 5.7, 7.8, 10.3, 12.9 eV going down the group, where ours stay level (F 3.401, Cl 3.613, Br 3.364 eV, measured). 80 of the 140 bind one by more than 0.7 eV, the level above which the method held in ours (appendix E.4); the light noble gases don't hold one at all.

6.5 The rows need the exponent 1/r. The rows survive the shell theorem, but not a change of exponent. Solved again under $1/r^{\eta}$ with $\eta = 0.5$ and 1.5, every pair of charges alike, elements 1–100 (figure 1c, appendix F.8), the table's noble gases move. Under $\eta = 0.5$ an electron with more angular momentum is the more bound, element 4 is $1s^2\ 2p^2$, each shell fills whole before the next begins, and the noble gases fall at 2, 12, 40 and 100: hydrogen's shells. Under $\eta = 1.5$ the next shell's s sinks below the current p (element 5 is $1s^2\ 2s^2\ 3s^1$, 14 is $1s^2\ 2s^2\ 2p^8\ 3s^2$), and they fall at 2, 14, 24, 34 and 72 (82 of 100 settle). By the same rule the pinned law's fall at 2, 12, 22, 42, 50, 70 and 78: its noble gases and two closed d-and-s atoms of zinc's kind. Only $1/r$ gives the rows the generalized Madelung rule predicts.

7 · Neutral atoms aren't silent

Under the shell theorem a neutral atom is invisible from outside its cloud. Here it isn't. A shell of charge q and radius s, felt at a distance R outside it, gives $q\,\kappa(s/R)/R$ with $\kappa = 1 - t^2/8 - \dots$ (appendix A). Summed over the nucleus and the electrons the leading terms cancel, and what remains, for an electron at distance R, is a pull $-\frac{Q}{8R^3}$ with $Q = \sum \langle r^2 \rangle$. The solved noble gases follow it: at 30 bohr the full cloud's potential and the far form agree to 0.22 %. Between two atoms with round clouds (closed shells, or a single s electron) the same failure gives a permanent attraction $-\frac{3Q_A Q_B}{64R^5}$, with no dipole needed, which outranges London's R^{-6}; an open p, d or f shell isn't round, and adds an ordinary quadrupole pull of the same order that depends on how the atoms are turned (appendix J.2).

For two like atoms the permanent pull's coefficient is $C_5 = 3Q^2/64$, and $R^* = C_6/C_5$ is the distance inside which dispersion is the larger, with C_6 from the solver's coupled response by Casimir and Polder's integral (appendix J.1, table 2; in ours its noble gases' polarizabilities are off by −8 % to +4 % against Derevianko, Porsev, and Babb (2010)). These are computed for 15 four-dimensional atoms, those whose clouds are round: for hydrogen and helium R^* is 28 and 9.26 bohr, for the alkali-like and alkaline-earth-like atoms sampled 4.9–94, and for the other 5 closed-shell atoms sampled 0.59–1.5, inside the atom. Among them, past contact, the permanent term is the leading pull at every distance where a far-field form applies. Pulled apart by Hartree–Fock, two closed-shell atoms follow it (1.02–1.06 times the far form at 16 bohr), so the atoms the molecular codes separate into are this paper's (appendix J.2).

That sets a requirement on every later model of four-dimensional matter. In ours the trace of a charge distribution's second moment is dropped from multipole expansions, because $1/r$ is harmonic and the trace carries no field. Here it is a neutral atom's leading field, so a four-dimensional force field has to keep it; the later papers' do, as a cloud term on every atom.

8 · Limitations and conclusion

Limitations. The force acts instantly (section 2.5), so only statics are safe, and nothing here moves. Heavy atoms would need relativity, which the law has given up for the force between charges, so the table past about element 100 is a first pass; round a point nucleus the innermost Dirac level fails at $Z \approx 205$ in four dimensions (Katsura and Aoki 2006; GU, MA, and DONG 2002), beyond the table. Nuclei are declared stable to $Z = 140$. The method has local exchange and no correlation, which in ours favours d by about a volt (appendix E.3), every subshell's cloud averaged over the sphere, and open subshells' terms resolved to first order on the solver's own orbitals. The counterfactual leaves row 6's f block unsettled. Nothing here claims that four-dimensional chemistry looks like ours; the next papers take up bonds, liquids and crystals under the same law.

Conclusion. A world with four large dimensions of space has to choose its law between charges, because Gauss's law gives it no atoms. Pinned at $1/r$, the only power law that keeps a shell together, it has atoms, and they have a periodic table: rows of 2, 10, 10, 28, 28, 60, noble gases at 2, 12, 22, 50, 78, 138, as the generalized Madelung rule predicts. The rows survive the shell theorem forced back on but not a change of exponent: they need $1/r$ itself. What fills inside them from the first d

element on doesn't survive the shell theorem, because in four dimensions $1/r$ isn't harmonic and an inner cloud screens only partly: outer electrons are held about two and a half times harder, the hardest-held electron belongs to a full d shell rather than a noble gas, and a neutral atom pulls from afar. And four-dimensional space breaks Hund's first rule for d^2, d^{16}, $4f^3$ and $4f^{29}$; for d^2, on element 26's orbital, Gauss's $1/r^2$ breaks it too.

Author contributions

minor mobius chose the question and the law between charges, judged the results and directed the work. Claude wrote the solvers and the tests, ran the calculations and wrote the text. Astra audited the claims, the references and the reproduction, in rounds whose findings and answers are on the paper's process page. Claude and Astra are AI systems.

Supplementary material

The supplement holds the notation and appendices [A](supplement.html#app-a)–K: the arguments and methods in full, every check against our own world, the starting data for chemistry, the atoms' response, and how to reproduce every number. Supplement S1 is the table with every open subshell given Hund spins (`data/table4-hundspin.csv`). The 42 claims this text rests on, each recomputed from the data behind it, are in `ledger.json` and on the paper's process page; the solvers, MIT-licensed, are at tangled.org/morphyxmino.bsky.social/four-atom, the interactive table at four.mino.mobi/table, and figure 5's hydrogen twin at kata.mino.mobi/orbital.

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